Counting Primes, Groups and Manifolds
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This is joint work with A. Lubotzky and includes results by D. Goldfeld and L. Pyber.

Let $\Lambda = \text{SL}_2(\mathbb{Z})$ be the modular group and let $c_n(\Lambda)$ be the number of congruence subgroups of $\Lambda$ of index at most $n$. We prove that $\lim_{n \to \infty} \frac{\log c_n(\Lambda)}{(\log n)^2 / \log \log n} = \frac{3 - 2\sqrt{2}}{2}$. The proof is based on the Bombieri-Vinogradov ‘Riemann hypothesis on the average’ and on the solution of a new type of extremal problem in combinatorial number theory. Similar surprisingly sharp estimates are obtained for the subgroup growth of lattices in higher rank semisimple Lie groups. If $G$ is such a Lie group and $\Gamma$ is an irreducible lattice of $G$ it turns out that the subgroup growth of $\Gamma$ is independent of the lattice and depends only on the Lie type of the direct factors of $G$. It can be calculated easily from the root system. The most general case of this result relies on the Generalized Riemann Hypothesis but many special cases are unconditional. The proofs use techniques from number theory, algebraic groups, finite group theory and combinatorics.